

On tiling the integers with 4-sets of the same gap sequence

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Abstract

Partitioning a set into similar, if not, identical, parts is a fundamental research topic in combinatorics. The question of partitioning the integers in various ways has been considered throughout history. Given a set $\{x_1, \dots, x_n\}$ of integers where $x_1 < \dots < x_n$, let the *gap sequence* of this set be the nondecreasing sequence d_1, \dots, d_{n-1} where $\{d_1, \dots, d_{n-1}\}$ equals $\{x_{i+1} - x_i : i \in \{1, \dots, n-1\}\}$ as a multiset. This paper addresses the following question, which was explicitly asked by Nakamigawa: can the set of integers be partitioned into sets with the same gap sequence? The question is known to be true for any set where the gap sequence has length at most two. This paper provides evidence that the question is true when the gap sequence has length three. Namely, we prove that given positive integers p and q , there is a positive integer r_0 such that for all $r \geq r_0$, the set of integers can be partitioned into 4-sets with gap sequence p, q, r .

1 Introduction

Let $[n]$ denote the set $\{1, \dots, n\}$ and let $[a, b]$ denote the set $\{a, \dots, b\}$. Note that $[1, 0] = \emptyset$. An n -set is a set of size n .

Partitioning a set into similar, if not, identical, parts is a fundamental research topic in combinatorics. In the literature, it is typically said that T *tiles* S if the set S can be partitioned into parts that are all “similar” to T in some sense. For example, Golomb initiated the study of tilings of the checker board with “polyominoes” in 1954 [Gol54], and it has attracted a vast audience of both mathematicians and non-mathematicians. See the book by Golomb [Gol94] for recent developments regarding this particular problem.

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The question of partitioning the integers \mathbb{Z} (and the positive integers \mathbb{Z}^+) in various ways has been considered throughout history. For two sets T and S where $T \subseteq S$ and a group G acting on S , we say that “ T tiles S under G ” if S can be partitioned into copies that are obtainable from T via G ; namely, there is a subset X of G such that $S = \coprod_{\gamma \in X} \gamma(T)$. Tilings of \mathbb{Z} and \mathbb{Z}^+ under translation have already been extensively studied [dB50, Lon67]. It is known that a set S of integers tiles \mathbb{Z}^+ under translation if and only if S tiles some interval of \mathbb{Z} under translation. In particular, a 3-set S tiles \mathbb{Z}^+ under translation if and only if the elements of S form an arithmetic progression.

It is easy to see that an arbitrary 2-set of integers tiles an interval of \mathbb{Z} (and therefore tiles \mathbb{Z}) under translation, and there are 3-sets of integers that do not tile \mathbb{Z} under translation. However, if both translation and reflection are allowed, then Sands and Swierczkowski [SS60] provided a short proof that an arbitrary 3-set of real numbers tiles \mathbb{R} (simplifying a proof in [KS58]), and on the way they also proved that an arbitrary 3-set of integers tiles \mathbb{Z} . It is also known that not all 4-sets of integers tile \mathbb{Z} under translation and reflection.

In his book [Hon76], Honsberger strengthened the previous result with a simple greedy algorithm by showing that an arbitrary 3-set of integers tiles an interval of \mathbb{Z} under translation and reflection. Meyerowitz [Mey88] analyzed this algorithm and gave a constructive proof that the algorithm produces a tiling of an interval of \mathbb{Z} , and also proved that a 3-set of real numbers tiles \mathbb{R}^+ , strengthening an aforementioned result. This algorithm does not necessarily find the shortest interval of \mathbb{Z} that a 3-set of integers can tile; there has been effort in trying to determine the shortest such interval [AH81, Nak00], and in some cases the shortest such interval is known.

Gordan [Gor80] generalized the problem to higher dimensions. He proved that a 3-set of \mathbb{Z}^n tiles \mathbb{Z}^n under the Euclidean group actions (translation, reflection, and rotation), and that there is a set of size $4n - 2\lfloor n/2 \rfloor$ of \mathbb{Z}^n that does not tile \mathbb{Z}^n under the Euclidean group actions. More information regarding higher dimensions is in Section 4. There is also a paper [Nak15] that studies tilings of the cyclic group \mathbb{Z}_n .

This paper focuses on partitioning \mathbb{Z} into sets with the same “gap sequence” and “gap length”, which is the term used in [Nak15] and [Nak05], respectively. Given a set $\{x_1, \dots, x_n\}$ of integers where $x_1 < \dots < x_n$, let the *gap sequence* of this set be the nondecreasing sequence d_1, \dots, d_{n-1} where $\{d_1, \dots, d_{n-1}\}$ equals $\{x_{i+1} - x_i : i \in \{1, \dots, n-1\}\}$ as a multiset. Note that the gap sequence of a set with n elements has length $n - 1$. Roughly speaking, in addition to reflecting the order of the gaps of a given set, any permutation of the order of the gaps of the set is allowed.

In [Nak05], the following question was explicitly asked:

Question 1.1 ([Nak05]). *For a gap sequence S of length $n - 1$, can \mathbb{Z} be partitioned into n -sets with the same gap sequence S ?*

Since allowing permutations of the order of the gaps of a given set does not provide additional help (when reflections of the gaps are already allowed), previous results above imply that this question is true when $n \in \{1, 2, 3\}$. In this paper, we prove the following theorem that provides evidence that the question is true when $n = 4$. Corollary 1.3 is an immediate consequence of the theorem.

Theorem 1.2. *There is an interval of the integers that can be partitioned into 4-sets with the same gap sequence p, q, r , if $q \geq p$ and $r \geq \max\{4q(4q - 1), \frac{1}{\gcd(p, q)}(5p + 4q - \gcd(p, q))(4p + 3q - \gcd(p, q))\}$.*

Corollary 1.3. *There is an interval of the integers that can be partitioned into 4-sets with the same gap sequence p, q, r , if $r \geq 63(\max\{p, q\})^2$.*

Note that for the sake of presentation, we omit some improvements on the constants of the threshold on r . Our proof follows the ideas in [Nak00, Nak05], where partitions of \mathbb{Z}^2 is used to aid the partition of \mathbb{Z} . We develop and push the method further and generalize it to \mathbb{Z}^3 . In Section 2, we show that we can partition certain subsets of \mathbb{Z}^3 into smaller subsets of \mathbb{Z}^3 that we call *blocks*. In Section 3, we demonstrate how to use the lemmas in Section 2 to tile an interval of \mathbb{Z} with 4-sets with the desired gap sequence. We finish the paper with some open questions in Section 4.

2 Lemmas

Given three vectors d_1, d_2, d_3 in \mathbb{Z}^3 , a 4-set $\{v_1, v_2, v_3, v_4\}$ of \mathbb{Z}^3 in which $\{v_4 - v_3, v_3 - v_2, v_2 - v_1\} = \{d_1, d_2, d_3\}$ is called a (d_1, d_2, d_3) -*block*. For a set V of triples of vectors in \mathbb{Z}^3 , we say a set S of \mathbb{Z}^2 can be *covered* (with *height* $h(S)$) by V -blocks if there exists an integer $h(S)$ such that $S \times h(S) = \{(x, y, z) : (x, y) \in S, z \in [h(S)]\}$ can be partitioned into blocks from V . If V only has one vector v , then we simply write “covered by v -blocks” instead of “covered by $\{v\}$ -blocks”.

Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$ be unit vectors in \mathbb{Z}^3 . By stretching $X \subset \mathbb{Z}^3$ in the e_1 , e_2 , and e_3 direction by a real number w , we obtain $\{(wx, y, z) : (x, y, z) \in X\}$, $\{(x, wy, z) : (x, y, z) \in X\}$, and $\{(x, y, wz) : (x, y, z) \in X\}$, respectively.

2.1 When $q \geq 2p$

Lemma 2.1. *The following sets of \mathbb{Z}^2 can be covered by (e_1, e_2, e_3) -blocks:*

- (i) $S_1 = \{(1, 1), (1, 2), (2, 2)\}$ with $h(S_1) = 4$
- (ii) $S_2 = \{(1, 1), (2, 1), (2, 2)\}$ with $h(S_2) = 4$
- (iii) $S_3 = [3] \times [2]$ with $h(S_3) = 4$
- (iv) $S_4 = [k] \times [4]$ for $k \geq 2$ with $h(S_4) = 20$
- (v) $S_5 = ([2] \times [4]) \cup \{(3, 1), (3, 2)\}$ with $h(S_5) = 4$
- (vi) $S_6 = ([2] \times [4]) \cup \{(3, 4)\}$ with $h(S_6) = 4$
- (vii) $S_7 = ([k] \times [4]) \cup \{(k + 1, 4)\}$ for $k \geq 2$ with $h(S_7) = 20$

Proof. See Figure 1 for an illustration of some cases.

(i):

$$\begin{aligned} B_1 &= \{(1, 1, 1), (1, 2, 1), (2, 2, 1), (2, 2, 2)\}, \\ B_2 &= \{(1, 1, 2), (1, 2, 2), (1, 2, 3), (2, 2, 3)\}, \\ B_3 &= \{(1, 1, 3), (1, 1, 4), (1, 2, 4), (2, 2, 4)\}. \end{aligned}$$

(ii):

$$\begin{aligned} B_1 &= \{(1, 1, 1), (2, 1, 1), (2, 2, 1), (2, 2, 2)\}, \\ B_2 &= \{(1, 1, 2), (2, 1, 2), (2, 1, 3), (2, 2, 3)\}, \\ B_3 &= \{(1, 1, 3), (1, 1, 4), (2, 1, 4), (2, 2, 4)\}. \end{aligned}$$

(iii): Combine S_1 and the block obtained by shifting S_2 by e_1 .

(iv): If k is even, then we can do better and obtain $h(S_4) = 5$. It is not hard to see we can fill S_4 with blocks of $[2] \times [4]$ by putting them side by side, so it is sufficient to show how to fill $[2] \times [4]$.

$$\begin{aligned} B_1 &= \{(1, 1, 1), (2, 1, 1), (2, 1, 2), (2, 2, 2)\}, \\ B_2 &= \{(1, 2, 1), (2, 2, 1), (2, 3, 1), (2, 3, 2)\}, \\ B_3 &= \{(1, 3, 1), (1, 4, 1), (2, 4, 1), (2, 4, 2)\}, \\ B_4 &= \{(1, 1, 2), (1, 2, 2), (1, 2, 3), (2, 2, 3)\}, \\ B_5 &= \{(1, 3, 2), (1, 4, 2), (1, 4, 3), (2, 4, 3)\}, \\ B_6 &= \{(1, 1, 3), (2, 1, 3), (2, 1, 4), (2, 2, 4)\}, \\ B_7 &= \{(1, 3, 3), (2, 3, 3), (2, 3, 4), (2, 4, 4)\}, \\ B_8 &= \{(1, 1, 4), (1, 1, 5), (2, 1, 5), (2, 2, 5)\}, \\ B_9 &= \{(1, 2, 4), (1, 2, 5), (1, 3, 5), (2, 3, 5)\}, \\ B_{10} &= \{(1, 3, 4), (1, 4, 4), (1, 4, 5), (2, 4, 5)\}. \end{aligned}$$

If k is odd, then $h(S_4) = 20$. It is not hard to see we can fill S_4 with two copies of S_3 (which was already shown to be covered in (iii)) by putting one on top of another and then using blocks of $[2] \times [4]$ side by side. Note that the least common multiple of $h(S_3) = 4$ and $h([2] \times [4]) = 5$ is 20.

(v):

$$\begin{aligned} B_1 &= \{(1, 1, 1), (1, 1, 2), (1, 2, 2), (2, 2, 2)\}, \\ B_2 &= \{(1, 2, 1), (2, 2, 1), (2, 3, 1), (2, 3, 2)\}, \\ B_3 &= \{(2, 1, 1), (3, 1, 1), (3, 2, 1), (3, 2, 2)\}, \\ B_4 &= \{(1, 3, 1), (1, 4, 1), (2, 4, 1), (2, 4, 2)\}, \\ B_5 &= \{(2, 1, 2), (3, 1, 2), (3, 1, 3), (3, 2, 3)\}, \\ B_6 &= \{(1, 3, 2), (1, 4, 2), (1, 4, 3), (2, 4, 3)\}, \\ B_7 &= \{(1, 1, 3), (1, 1, 4), (1, 2, 4), (2, 2, 4)\}, \\ B_8 &= \{(1, 2, 3), (2, 2, 3), (2, 3, 3), (2, 3, 4)\}, \\ B_9 &= \{(2, 1, 3), (2, 1, 4), (3, 1, 4), (3, 2, 4)\}, \\ B_{10} &= \{(1, 3, 3), (1, 3, 4), (1, 4, 4), (2, 4, 4)\}. \end{aligned}$$

(vi):

$$\begin{aligned}
B_1 &= \{(1, 1, 1), (2, 1, 1), (2, 2, 1), (2, 2, 2)\}, \\
B_2 &= \{(1, 2, 1), (1, 2, 2), (1, 3, 2), (2, 3, 2)\}, \\
B_3 &= \{(1, 3, 1), (1, 4, 1), (1, 4, 2), (2, 4, 2)\}, \\
B_4 &= \{(2, 3, 1), (2, 4, 1), (3, 4, 1), (3, 4, 2)\}, \\
B_5 &= \{(1, 1, 2), (2, 1, 2), (2, 1, 3), (2, 2, 3)\}, \\
B_6 &= \{(1, 1, 3), (1, 1, 4), (2, 1, 4), (2, 2, 4)\}, \\
B_7 &= \{(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)\}, \\
B_8 &= \{(1, 3, 3), (1, 4, 3), (1, 4, 4), (2, 4, 4)\}, \\
B_9 &= \{(2, 3, 3), (2, 4, 3), (3, 4, 3), (3, 4, 4)\}.
\end{aligned}$$

(vii): Assume k is even. It is not hard to see we can fill S_7 with one S_6 (which was already shown to be covered in (vi)) and then using blocks of $[2] \times [4]$ side by side. Note that the least common multiple of $h(S_6) = 4$ and $h([2] \times [4]) = 5$ is 20.

Assume k is odd. When $k = 3$, it is not hard to see that we can fill S_7 with one S_1 (which was already shown to be covered in (i)) and one S_5 (which was already shown to be covered in (v)). When $k > 3$, attach blocks of $[2] \times [4]$ side by side to the configuration when $k = 3$. Note that the least common multiple of $h(S_6) = 4$ and $h([2] \times [4]) = 5$ is 20. \square

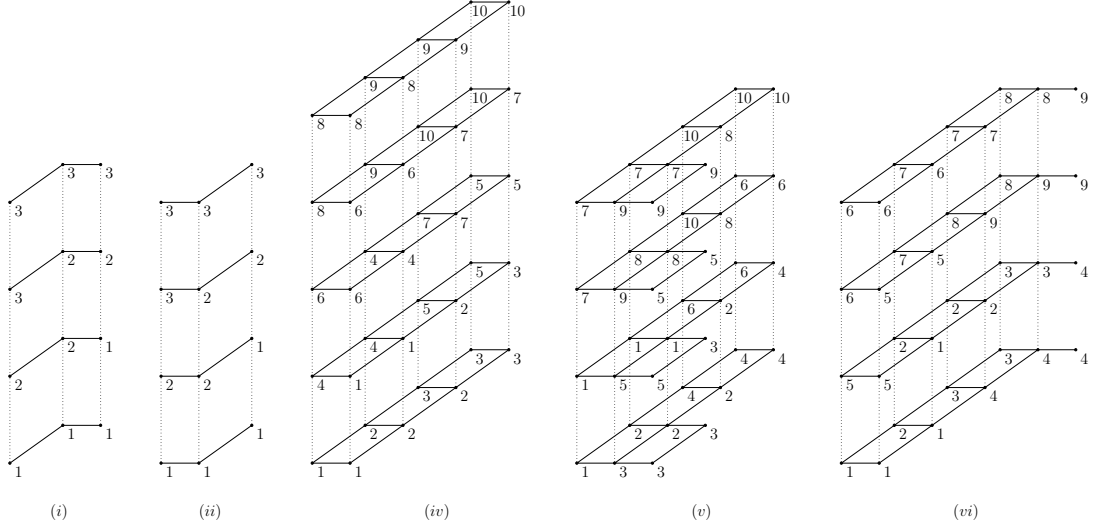


Figure 1: Illustration for some cases of Lemma 2.1.

Lemma 2.2. *Given $q \geq 2p$, the following sets can be covered by (pe_1, e_2, e_3) -blocks:*

- (i) $X_1 = [q] \times [4]$ with $h(X_1) = 20$
- (ii) $X_2 = ([q] \times [4]) \cup \{(q+1, 4)\}$ with $h(X_2) = 20$

Proof. It is sufficient to show that X_1 and X_2 can be covered by (e_1, e_2, e_3) -blocks, but stretched by p .

Let $a = \lfloor \frac{q}{p} \rfloor$ and $b = q - ap$ so that $b \in [0, p-1]$. Note that $a \geq 2$ since $q \geq 2p$. Obtain P_4^{a+1} , P_4^a , and P_7^a by stretching S_4 with $k = a+1$, S_4 with $k = a$, and S_7 with $k = a$, respectively, from Lemma 2.1 in the e_1 direction by p ; in other words, $P_4^{a+1} = \{(1+ip, y) : i \in [0, a], y \in [4]\}$, $P_4^a = \{(1+ip, y) : i \in [0, a-1], y \in [4]\}$, and $P_7^a = P_4^a \cup \{(1+ap, 4)\}$.

Let $P_4^* = \{P_4^{a+1} + (i, 0) : i \in [0, b-1]\}$ and $P_4^{**} = \{P_4^a + (i, 0) : i \in [b+1, p-1]\}$.

(i): Now P_4^* , $P_4^a + (b, 0)$, P_4^{**} is a partition of X_1 . Since S_4 can be covered with height 20, we conclude that X_1 can be covered with height 20.

(ii): Now P_4^* , $P_7^a + (b, 0)$, P_4^{**} is a partition of X_2 . Since S_4 and S_7 can be covered with height 20, we conclude that X_2 can be covered with height 20.

See Figure 2 for an illustration. □

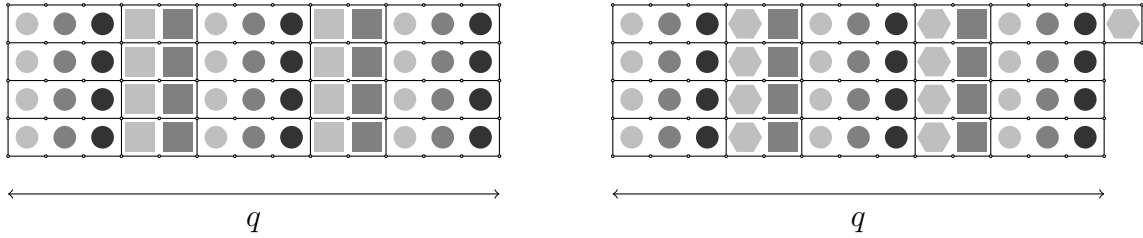


Figure 2: Figure for Lemma 2.2. Same shape and same shade means same block. Same shape and different shade means same type of block, but different block.

2.2 When $p \leq q \leq 2p$

Lemma 2.3. *The following sets of \mathbb{Z}^2 can be covered by $(e_1, e_2 - e_1, e_3)$ -blocks:*

- (i) $T_1 = \{(1, 1), (1, 2), (2, 1)\}$ with $h(T_1) = 4$
- (ii) $T_2 = \{(1, 2), (2, 1), (2, 2)\}$ with $h(T_2) = 4$
- (iii) $T_3 = \{(1, 2), (1, 3), (2, 1), (2, 2)\}$ with $h(T_3) = 2$
- (iv) $T_4 = [1, 2] \times [1, 2]$ with $h(T_4) = 2$
- (v) $T_5 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\}$ with $h(T_5) = 2$

Proof. See Figure 3 for an illustration.

(i):

$$\begin{aligned} B_1 &= \{(1, 1, 1), (2, 1, 1), (1, 2, 1), (1, 2, 2)\}, \\ B_2 &= \{(1, 1, 2), (2, 1, 2), (2, 1, 3), (1, 2, 3)\}, \\ B_3 &= \{(1, 1, 3), (1, 1, 4), (2, 1, 4), (1, 2, 4)\}. \end{aligned}$$

(ii):

$$\begin{aligned} B_1 &= \{(2, 1, 1), (1, 2, 1), (2, 2, 1), (2, 2, 2)\}, \\ B_2 &= \{(2, 1, 2), (1, 2, 2), (1, 2, 3), (2, 2, 3)\}, \\ B_3 &= \{(2, 1, 3), (2, 1, 4), (1, 2, 4), (2, 2, 4)\}. \end{aligned}$$

(iii):

$$\begin{aligned} B_1 &= \{(2, 1, 1), (2, 1, 2), (1, 2, 2), (2, 2, 2)\}, \\ B_2 &= \{(1, 2, 1), (2, 2, 1), (1, 3, 1), (1, 3, 2)\}. \end{aligned}$$

(iv):

$$\begin{aligned} B_1 &= \{(1, 1, 1), (1, 1, 2), (2, 1, 2), (1, 2, 2)\}, \\ B_2 &= \{(2, 1, 1), (1, 2, 1), (2, 2, 1), (2, 2, 2)\}. \end{aligned}$$

(v):

$$\begin{aligned} B_1 &= \{(1, 1, 1), (1, 1, 2), (2, 1, 2), (1, 2, 2)\}, \\ B_2 &= \{(2, 1, 1), (3, 1, 1), (3, 1, 2), (2, 2, 2)\}, \\ B_3 &= \{(1, 2, 1), (2, 2, 1), (1, 3, 1), (1, 3, 2)\}. \end{aligned}$$

□

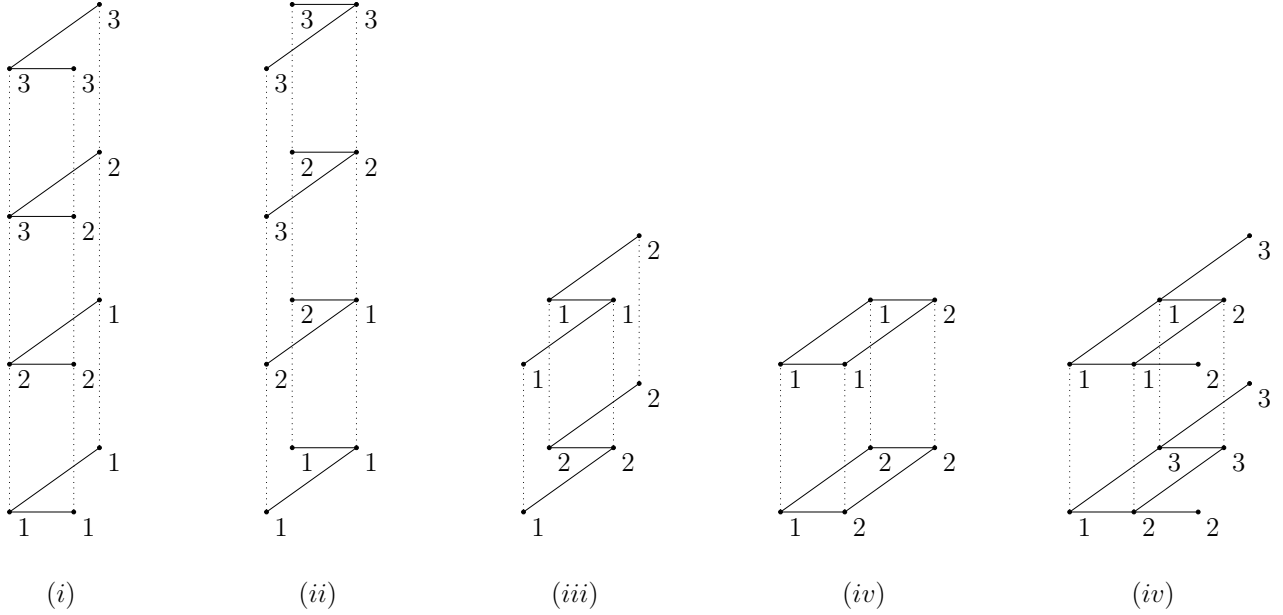


Figure 3: Illustration for Lemma 2.3.

Lemma 2.4. *Given $p \leq q \leq 2p$, the following sets can be covered by $\{(pe_1, e_2 - pe_1, e_3), (qe_1, e_2 - qe_1, e_3)\}$ -blocks:*

(i) $Y_1 = ([p+q] \times [4]) \cup \{(x, 5) : x \in [p]\}$ with $h(Y_1) = 4$

(ii) $Y_2 = ([p+q] \times [3]) \cup \{(x, 4) : x \in [p]\}$ with $h(Y_2) = 4$

Proof. It is sufficient to show that Y_1 and Y_2 can be covered by $(e_1, e_2 - e_1, e_3)$ -blocks, but stretched by either p or q .

Let $q = p + t$ so that $t \in [0, p]$. Obtain P_1, P_2, P_3, P_4 , and P_5 by stretching T_1, T_2, T_3, T_4 , and T_5 respectively, from Lemma 2.3 in the e_1 direction by p ; in other words $P_1 = \{(1, 1), (p+1, 1), (1, 2)\}$, $P_2 = \{(1, 2), (p+1, 1), (p+1, 2)\}$, $P_3 = \{(1, 2), (1, 3), (1+p, 1), (1+p, 2)\}$, $P_4 = \{(1, 1), (1, 2), (1+p, 1), (1+p, 2)\}$, $P_5 = \{(1, 1), (1, 2), (1, 3), (1+p, 1), (1+p, 2), (1+2p, 1)\}$. Obtain Q_1 by stretching T_1 from Lemma 2.3 in the e_1 direction by q ; in other words $Q_1 = \{(1, 1), (q+1, 1), (1, 2)\}$.

(i): Let $P_1^* = \{P_1 + (i, 0) : i \in [t, p-1]\}$, $P_2^* = \{P_2 + (i, 1) : i \in [t, p-1]\}$, $P_3^* = \{P_3 + (i, 1) : i \in [p, p+t-1]\}$, $P_5^* = \{P_5 + (i, 0) : i \in [0, t-1]\}$, and $Q_1^* = \{Q_1 + (i, 3) : i \in [0, p-1]\}$. Now $P_1^*, P_2^*, P_3^*, P_5^*, Q_1^*$ is a partition of Y_1 .

(ii): Let $P_1^{**} = \{P_1 + (i, 0) : i \in [0, t-1]\}$, $P_3^{**} = \{P_3 + (i, 0) : i \in [p, p+t-1]\}$, $P_4^{**} = \{P_4 + (i, 0) : i \in [t, p-1]\}$, and $Q_1^{**} = \{Q_1 + (i, 2) : i \in [0, p-1]\}$. Now $P_1^{**}, P_3^{**}, P_4^{**}, Q_1^{**}$ is a partition of Y_2 .

See Figure 4 for an illustration. □

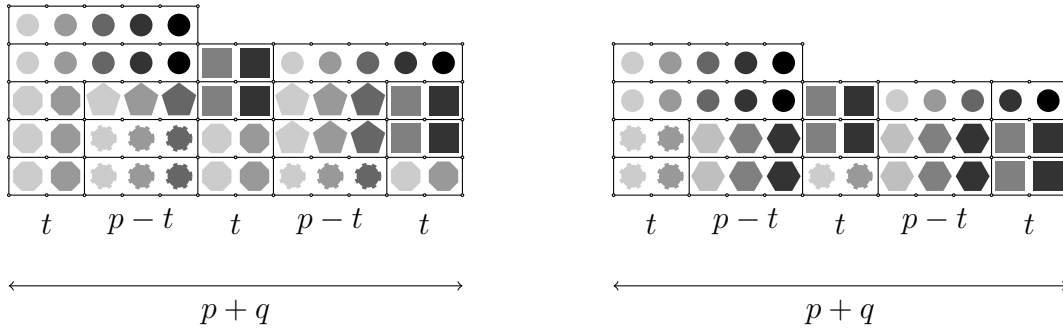


Figure 4: Figure for Lemma 2.4. Same shape and same shade means same block. Same shape and different shade means same type of block, but different block.

3 Main result

A set $S \subset \mathbb{Z}^2$ is called a *layer*, and S is an *a-nice layer* if S is of the form $([a] \times [b]) \cup \{(i, b+1) : i \in [c]\}$ where a, b, c are integers. Given an element $(x, y) \in S$, the *row* of (x, y) is the set of elements in S with the same second coordinate. Note that the sets X_1 and X_2 from Lemma 2.2 are q -nice layers and the sets Y_1 and Y_2 from Lemma 2.4 are $(p+q)$ -nice layers.

For convenience, we will say a set with gap sequence d_1, \dots, d_{n-1} is a $(d_{\sigma(1)}, \dots, d_{\sigma(n-1)})$ -set for any permutation σ of $[n-1]$.

Lemma 3.1. *For each $i \in [n]$, let S_i be an a -nice layer that can be covered with height $h(S_i)$ by V -blocks, and let $l = \text{lcm}_{i \in [n]} \{h(S_i)\}$. For $r \geq 1 - d + d \sum_{i \in [n]} |S_i|$ and positive integers d, p, q with $q \geq p$, the set $\bigcup_{j \in [l]} (d[\sum_{i \in [n]} |S_i|] + (j-1)r)$ can be partitioned into*

- (i) (dp, da, r) -sets when $V = \{(pe_1, e_2, e_3)\}$.
- (ii) $(dp, d(a-p), r)$ -sets and $(dq, d(a-q), r)$ -sets when $V = \{(pe_1, e_2 - pe_1, e_3), (qe_1, e_2 - qe_1, e_3)\}$.

Proof. Let $<$ be an ordering of the elements of $\{(S_i \times l, i) : i \in [n]\}$ such that $((x_1, y_1, z_1), i_1) < ((x_2, y_2, z_2), i_2)$ if (a) $z_1 < z_2$ or (b) $z_1 = z_2$ and $i_1 < i_2$ or (c) $z_1 = z_2$, $i_1 = i_2$, and $y_1 < y_2$ or (d) $z_1 = z_2$, $i_1 = i_2$, $y_1 = y_2$, and $x_1 < x_2$. This ordering $<$ gives a natural bijection φ between $\{(S_i \times l, i) : i \in [n]\}$ and $\bigcup_{j \in [l]} (d[\sum_{i \in [n]} |S_i|] + (j-1)r)$. Note that the condition $r \geq 1 - d + d \sum_{i \in [n]} |S_i|$ is needed to ensure that φ is a bijection.

(i) Assume each S_i can be covered by (pe_1, e_2, e_3) -blocks, and let u and v be two elements of one particular block. If $|u - v| = pe_1$, then $|\varphi(u) - \varphi(v)| = dp$ since u and v are in the same row. If $|u - v| = e_2$, then $|\varphi(u) - \varphi(v)| = da$ since the lower row of u and v has exactly a elements. If $|u - v| = e_3$, then $|\varphi(u) - \varphi(v)| = r$ since u and v must be in different layers.

Therefore, $\bigcup_{j \in [l]} (d[\sum_{i \in [n]} |S_i|] + (j-1)r)$ can be partitioned into (dp, da, r) -sets.

(ii) Assume each S_i can be covered by $\{(pe_1, e_2 - pe_1, e_3), (qe_1, e_2 - qe_1, e_3)\}$ -blocks, and let u and v be two elements of a $(pe_1, e_2 - pe_1, e_3)$ -block. If $|u - v| = pe_1$, then $|\varphi(u) - \varphi(v)| = dp$ since u and v are in the same row. If $|u - v| = e_2 - pe_1$, then $|\varphi(u) - \varphi(v)| = d(a-p)$ since the lower row of u and v has exactly a elements. If $|u - v| = e_3$, then $|\varphi(u) - \varphi(v)| = r$ since u and v must be in different layers. The case when u and v are two elements of a $(qe_1, e_2 - qe_1, e_3)$ -block is analogous.

Therefore, $\bigcup_{j \in [l]} (d[\sum_{i \in [n]} |S_i|] + (j-1)r)$ can be partitioned into $(dp, d(a-p), r)$ -sets and $(dq, d(a-q), r)$ -sets. \square

Lemma 3.2. *Given positive integers r_1 and r_2 , let $d = \gcd(r_1, r_2)$, and also let p and q be positive integers such that $q \geq p$ and p/d and q/d are also integers. For all integers $r \geq d(r_1/d - 1)(r_2/d - 1)$, there is an interval of \mathbb{Z} that can be partitioned into (p, q, r) -sets if L_1 and L_2 is a layer of size r_1/d and r_2/d , respectively, and both L_1 and L_2 are*

- (i) (q/d) -nice layers that can be covered by $(pe_1/d, e_2, e_3)$ -blocks.
- (ii) $(p/d + q/d)$ -nice layers that can be covered by $\{(pe_1/d, e_2 - pe_1/d, e_3), (qe_1/d, e_2 - qe_1/d, e_3)\}$ -blocks.

Proof. Let $l = \text{lcm}\{h(L_1), h(L_2)\}$. An integer s is good if $(r_1/d - 1)(r_2/d - 1) \leq s \leq \frac{r-1+d}{d}$. Note that a good integer s satisfies $r \geq 1 - d + ds$. Since r_1/d and r_2/d are coprime, a good s can be expressed as a linear combination of r_1/d and r_2/d with nonnegative coefficients. Therefore, given a good s , the set $T(s) = \bigcup_{j \in [l]} (d[s] + (j-1)r)$ can be partitioned into (p, q, r) -sets by Lemma 3.1 since (p, q, r) -sets are also (q, p, r) -sets, for both (i) and (ii).

Let $r' = r - d\lfloor r/d \rfloor$ so that $r' \in [0, d-1]$. If $r' \neq 0$, then both $\lfloor \frac{r}{d} \rfloor$ and $\lfloor \frac{r}{d} \rfloor + 1$ are good, and therefore by the above paragraph, both $T(\lfloor \frac{r}{d} \rfloor)$ and $T(\lfloor \frac{r}{d} \rfloor + 1)$ can be partitioned into (p, q, r) -sets. Now, $\bigcup_{i \in [r']} (T(\lfloor \frac{r}{d} \rfloor + 1) + i) \cup \bigcup_{i \in [r'+1, d-1]} (T(\lfloor \frac{r}{d} \rfloor) + i) = [lr] + d$.

If $r' = 0$, then $\lfloor \frac{r}{d} \rfloor$ is good, and therefore by the first paragraph, $T(\lfloor \frac{r}{d} \rfloor)$ can be partitioned into (p, q, r) -sets. Now, $\bigcup_{i \in [1, d-1]} (T(\lfloor \frac{r}{d} \rfloor) + i) = [lr] + d$.

In both cases, $[lr] + d$ can be partitioned into (p, q, r) -sets. \square

Theorem 3.3. *For positive integers p, q with $q \geq 2p$, if $r \geq 4q(4q - 1)$, then there is an interval of \mathbb{Z} that can be partitioned into 4-sets of the same gap sequence p, q, r .*

Proof. Since $q \geq 2p$, the layer X_1 and X_2 in Lemma 2.2 is a q -nice layer of size $4q$ and $4q + 1$, respectively, that can be covered by (pe_1, e_2, e_3) -blocks. Note that $\gcd(4q, 4q + 1) = 1$. Thus, by Lemma 3.2, there is an interval of \mathbb{Z} that can be partitioned into (p, q, r) -sets for all integers $r \geq 4q(4q - 1)$. \square

Theorem 3.4. *For positive integers p, q with $q \in [p, 2p]$, if $r \geq \frac{1}{\gcd(p, q)}(5p + 4q - \gcd(p, q))(4p + 3q - \gcd(p, q))$, then there is an interval of \mathbb{Z} that can be partitioned into 4-sets of the same gap sequence p, q, r .*

Proof. Since $q \in [p, 2p]$, the layer Y_1 and Y_2 in Lemma 2.4 is a $(p + q)$ -nice layer of size $5p + 4q$ and $4p + 3q$, respectively, that can be covered by $\{(pe_1, e_2 - pe_1, e_3), (qe_1, e_2 - qe_1, e_3)\}$ -blocks. Note that $\gcd(5p + 4q, 4p + 3q) = \gcd(p, q)$. Thus, by Lemma 3.2, there is an interval of \mathbb{Z} that can be partitioned into (p, q, r) -sets for all integers $r \geq \gcd(p, q)(\frac{5p + 4q}{\gcd(p, q)} - 1)(\frac{4p + 3q}{\gcd(p, q)} - 1)$. \square

Theorem 1.2 follows directly from Theorem 3.3 and Theorem 3.4.

4 Future directions and open questions

As noted in the introduction, we omit some improvements on the constants of the threshold on r in Theorem 1.2. For example, it is not hard to show that $([q] \times [4]) \cup \{(q + j, 4) : j \in [i]\}$ can be covered by (pe_1, e_2, e_3) -blocks for all $i \in [0, p]$, but we only provided the proof when $i \in \{0, 1\}$. Finding more blocks in Lemma 2.1 and Lemma 2.3 will help finding more layers that can be covered in Lemma 2.2 and Lemma 2.4, and appropriate combinations will improve the constants on the threshold on r .

We approached Question 1.1 with the mind set of allowing all gap sequences, but focusing on the case when $n = 4$, which is the first open case. Another approach is to investigate the question for all n , but for special gap sequences. The following conjecture was explicitly made in [Nak05]:

Conjecture 4.1 ([Nak05]). *There is an interval of \mathbb{Z} that can be partitioned into $(k + l + 1)$ -sets with the same gap sequence $p_1, \dots, p_k, q_1, \dots, q_l$ where $p_1 = \dots = p_k$ and $q_1 = \dots = q_l$.*

The truth of Question 1.1 when $n = 3$ is equivalent to this conjecture when $k = l = 1$. Some partial results on this conjecture were made in [Nak05].

As mentioned in the introduction, Gordon [Gor80] investigated the question in higher dimensions. We iterate some open questions for the 2-dimensional case. As it is known that there is a 6-set of \mathbb{Z}^2 that does not tile \mathbb{Z}^2 under the Euclidean group actions, the following statement is stated as “conceivable” in [Gor80]:

Question 4.2 ([Gor80]). *Does every set S of \mathbb{Z}^2 with $|S| \leq 5$ tile \mathbb{Z}^2 under the Euclidean group actions?*

Gordon [Gor80] also proved that a 3-set of \mathbb{Z}^2 tiles $\mathbb{Z}^+ \times \mathbb{Z}$ under the Euclidean group actions, whereas there is a 4-set of \mathbb{Z}^2 that does not. Actually, the same 4-set does not even tile $\mathbb{Z}^+ \times \mathbb{Z}^+$ under the Euclidean group actions, but Gordon [Gor80] proved that every 2-set does tile $\mathbb{Z}^+ \times \mathbb{Z}^+$ under the Euclidean group actions. To the authors' knowledge, the following question, which appeared in [Gor80], is still open:

Question 4.3 ([Gor80]). *Does every 3-set of \mathbb{Z}^2 tile $\mathbb{Z}^+ \times \mathbb{Z}^+$ under the Euclidean group actions?*

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